On Pebble Motion on Graphs and Abstract Multi-robot Path Planning

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Abstract

A problem of rearranging a group of robots that are moving in a certain environment is addressed in this paper (multi-robot path planning). A case when a graph modeling the environment is bi-connected is particularly studied. The paper puts into a relation the well known problems of moving pebbles on graphs (sliding box puzzles) with problems of multi-robot path planning. Theoretical results gained for problems of pebble motion on graphs are utilized for the development of algorithms for multi-robot path planning. As the optimization variant of both problems (a shortest solution is required) is known to be computationally hard (*NP*-hard), we concentrate on construction of sub-optimal solving procedures. However, the quality of solution is still an objective. A process of a composition of a sub-optimal solution of the problem of multi-robot path planning (a plan) of the pre-calculated optimal plans for the sub-problems (macros) is suggested. The plan composition using macros was integrated into two existing sub-optimal solving algorithms. In both cases, substantial improvements of the quality of resulting plans were achieved in comparison to the original versions. The no less important result is that one of the existing algorithms was generalized by integrating macros for a larger class of problems of multi-robot path planning.

Introduction and Motivation

Consider a group of robots moving in a certain environment where each robot needs to reach a certain goal position. The condition that must be preserved during robot motion is that robots must avoid obstacles and they must not collide with each other. This problem ranks among the most challenging problems of artificial intelligence and it motivates efforts of theorists as well as practicians (Russell & Norvig, 2003). The main difficulty of the problem arises from the requirement on the optimality of solutions and from complex interactions among robots.

The primary motivations for the problem are tasks of moving objects in tight space. These tasks include rearranging containers in storage yards, coordination of movements of a large group of automated agents, or optimization of dense traffic. However, this is not the only motivation. Many tasks from virtual spaces can be also viewed as problems of path planning for multiple robots. An example may be data transfer with limited buffers at communication nodes, a coordination of a group of agents in strategic computer games, or planning movements in mass scenes in computer-generated imagery. The important feature we need to preserve in solving techniques for the problem is that the group of robots should be treated as a single entity – intuitively, keeping this in mind allow to produce solutions of higher quality.

We observed a similarity between the formal definition of the problem of *multi-robot path planning* as it is discussed in (Ryan, 2007) and the *problem of pebble motion on a graph* (Kornhauser *et al.*, 1984). As there is lot of theoretical results for pebble motion on graphs we tried to utilize these results in our algorithms.

Both studied problems are computationally difficult when the shortest possible solution is required (*NP-hard*) (Ratner & Warmuth, 1986). Therefore we concentrate on developing of sub-optimal methods. An approach described in this paper is based on the use of pre-calculated optimal solutions to sub-problems (*macros*). We successfully tried to integrate macros within two existing sub-optimal algorithms. The resulting solving process proved to be better in terms of length of solutions (shorter solutions are preferred) as well as in terms of runtime.

The main contributions of this paper consist in the following aspects: **(i)** the problem of multi-robot path planning and the problem of pebble motion on a graph are put into relation, **(ii)** two existing algorithms are improved by integration of macro utilization – one of the described algorithms is now one of the best algorithms for certain class of problems, and **(iii)** this algorithm was also extended so it is now applicable to more general class of problems.

Pebble Motion and Multi-robot Path Planning

Consider an *environment* with a group of mobile robots. The problem being addressed in this paper consists in finding paths for the group of robots that need to reach certain goal positions starting from the given initial positions. The robots **must not collide** with each other and they must

avoid obstacles in the environment.

A relatively strong abstraction is adopted in this paper. The environment where the robots are moving is modeled as an undirected graph. The vertices of the graph represent positions in the environment and the edges model an unblocked way from one vertex to another. The *time* is discrete in this abstraction; it is an infinite linearly ordered set isomorphic to $\{0,1,2,\ldots\}$ where each element is called a *time step*.

Formal Definitions of the Problems

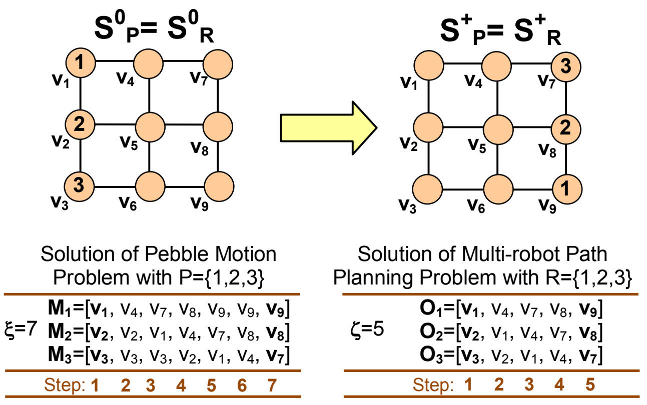
The following two definitions formalizes a problem of *pebble motion on a graph* (also called a pebble motion puzzle or sliding box puzzle) (Wilson, 1974; Kornhauser *et al.*, 1984) and the related problem of *path planning for multiple robots* (multi-robot path planning) (Ryan, 2007).

Definition 1 ***(problem of pebble motion on a graph).*** Let us have an undirected graph $G=(V,E)$. Next, let us have a set of pebbles $P=\{p_1,p_2,\ldots,p_\mu\}$ where $\mu<|V|$. The ***initial*** arrangement of the pebbles is defined by a simple function $S_P^0: P \rightarrow V$, that is $S_P^0(p_i) \neq S_P^0(p_j)$ for $i,j=1,2,\ldots,\mu$ with $i \neq j$. The ***goal*** arrangement of the pebbles is defined by a simple function $S_P^+: P \rightarrow V$, that is $S_P^+(p_i) \neq S_P^+(p_j)$ for $i,j=1,2,\ldots,\mu$ with $i \neq j$. The problem of ***pebble motion on a graph*** is a task to find a number ξ and a sequence of motions represented as a sequence of vertices $M_p=[m_1^p,m_2^p,\ldots,m_\xi^p]$ for every pebble $p \in P$ where $m_i^p \in V$ for $i=1,2,\ldots,\xi$, $m_1^p=S_P^0(p)$, $m_l^r=S_P^+(p)$, and either $\{m_i^p,m_{i+1}^p\} \in E$ or $m_i^p=m_{i+1}^p$ for $i=1,2,\ldots,$ $\xi-1$. Furthermore, sequences of motions $M_p=[m_1^p,$ $m_2^p,\ldots,m_\xi^p]$ and $M_q=[m_1^q,m_2^q,\ldots,m_\xi^q]$ for every two pebbles $p \in P$ and $q \in P$ such that $p \neq q$ must satisfy that $m_{i+1}^p \neq m_i^q$ for $i=1,2,\ldots,\xi-1$ (the target vertex must be unoccupied) and $m_i^p \neq m_i^q$ for $i=1,2,\ldots,\xi$ (no other pebble can simultaneously enter the target vertex). □

A problem of multi-robot path planning is a **relaxation** of the problem of pebble motion on a graph. The condition that the target vertex for a moving pebble/robot must be freed in the previous time step is relaxed. A motion of a robot entering the target vertex that is simultaneously left by another robot is allowed in multi-robot path planning. The problem is formalized in the following definition.

Definition 2 ***(problem of multi-robot path planning).*** Again, let us have an undirected graph $G=(V,E)$ but now instead of pebbles, a set of robots $R=\{r_1,r_2,\ldots,r_\mu\}$ where $\mu<|V|$ is given. The ***initial*** arrangement of the robots is defined by a simple function $S_R^0: R \rightarrow V$, that is $S_R^0(r_i) \neq S_R^0(r_j)$ for $i,j=1,2,\ldots,\mu$ with $i \neq j$. The ***goal*** arrangement of the robots is defined by a simple function $S_R^+: R \rightarrow V$, that is $S_R^+(r_i) \neq S_R^+(r_j)$ for $i,j=1,2,\ldots,\mu$ with $i \neq j$. The problem of ***multi-robot path planning*** is a task to find a number ζ and a sequence positions $O_r=[o_1^r,o_2^r,\ldots,o_\zeta^r]$ for every robot $r \in R$ where $o_i^r \in V$ for $i=1,2,\ldots,\zeta$, $o_1^r=S_R^0(r)$, $o_k^r=S_R^+(r)$, and either $\{o_i^r,o_{i+1}^r\} \in E$ or $o_i^r=o_{i+1}^r$ for $i=1,2,\ldots,\zeta-1$. Furthermore, sequences of positions $O_r=[o_1^r,o_2^r,\ldots,o_\zeta^r]$ and $O_s=[o_1^s,o_2^s,\ldots,o_\zeta^s]$ for every two robots $r \in R$ and $s \in R$ such that $r \neq s$ must satisfy that $o_i^r \neq o_i^s$ for $i=1,2,\ldots,\zeta$ (no two robots are simultaneously entering the same vertex). □

Both problems and their solutions are illustrated in figure 1. Observe different levels of parallelism.



$S^0_P = S^0_R$
$S^+_P = S^+_R$
v_1
v_2
v_3
v_4
v_5
v_6
v_7
v_8
v_9
1
2
3
Solution of Pebble Motion Problem with P={1,2,3}
ξ=7
$M_1=[v_1, v_4, v_7, v_8, v_9, v_9, v_9]$
$M_2=[v_2, v_2, v_1, v_4, v_7, v_8, v_8]$
$M_3=[v_3, v_3, v_3, v_2, v_1, v_4, v_7]$
Step: 1 2 3 4 5 6 7
Solution of Multi-robot Path Planning Problem with R={1,2,3}
ζ=5
$O_1=[v_1, v_4, v_7, v_8, v_9]$
$O_2=[v_2, v_1, v_4, v_7, v_8]$
$O_3=[v_3, v_2, v_1, v_4, v_7]$
Step: 1 2 3 4 5

Figure 1. *An illustration of problems of* ***pebble motion on a graph*** *and* ***multi-robot path planning***. The task is to move pebbles/robots from their initial positions specified by S_P^0 / S_R^0 to the goal positions specified by S_P^+ / S_R^+. A solution of length 7 is shown for the problem of pebble motion on a graph and a solution of length 5 is shown for the problem of multi-robot path planning. Notice the differences in parallelism between both solutions – multi-robot path planning allows a higher number of moves to be performed in parallel thanks to weaker requirements on solutions.

Summary of the Basic Properties of the Problems

Let us now summarize several basic properties of solutions of problems of pebble motion on graphs and multi-robot path planning.

Notice that problem of pebble motion on a graph as well as the problem of multi-robot path planning allows a pebble/robot to stay in a vertex for more than a single time step within the solution. It is also possible that a pebble/robot may visit the same vertex several times within the solution. Notice further that both problems intrinsically allow **parallel** movements of pebbles/robots. That is, more than one pebble/robot can move in a single time step. However, multi-robot path planning allows higher motion parallelism due to its weaker requirements (the target vertex is not required to be unoccupied in the previous time step before it is entered – see figure 1). To have a parallelism in the problem of pebble motion in a graph, more than one unoccupied vertex is necessary. On the other hand, it is sufficient to have a single unoccupied vertex to obtain parallelism in the solution of multi-robot path planning (consider for example robots moving around a cycle).

It is not difficult to observe that a solution to an instance of the problem of pebble motion on a graph is also a solution to the corresponding multi-robot path planning problem.

There is a variety of modifications of the defined problems. A natural additional requirement is to produce a shortest possible solution (that is, we require the numbers ξ or ζ respectively to be as small as possible). Unfortunately, this requirement makes the problem intractable (namely *NP*-hard; (Ratner & Warmuth, 1986)) while without the requirement both problems are in the *P* class (Kornhauser *et al.*, 1984). Nevertheless, we are usually concerned about the length of the solution in the real life situations. Taking into account the fact that existing fast sub-optimal methods (Kornhauser *et al.*, 1984) generate too long solutions, we need some alternative sub-optimal solving method.

All the algorithms developed in the following sections are designed for the problem of pebble motion on a graph. This is without loss of generality, since we know that algorithms for pebble motion on a graph apply also for multi-robot path planning. The parallelism within the solution of the multi-robot path planning problem can be increased in a post-processing step using a *critical path method*. Nevertheless, this issue is out of scope of this paper, for further details we refer the reader to (Russell & Norvig, 2003; Surynek, 2009c).

A Special Case with Bi-connected Graph

A special case of the problem is addressed in this paper. A case where the graph modeling the environment is bi-connected and there is only one unoccupied vertex (that is, $\mu = |V| - 1$) is studied.

Graph Theoretical Preliminaries

Let us recall some graph theoretical notions (West, 2000) that represent foundations for algorithms presented further.

Definition 3 ***(graph connectivity).*** An undirected graph $G = (V, E)$ is *connected*, if $|V| \geq 2$ and for every pair of distinct vertices $u, v \in V$ there is a path connecting u and v consisting of edges from E. □

Definition 4 ***(graph bi-connectivity).*** An undirected graph $G = (V, E)$ is *bi-connected*, if $|V| \geq 3$ and the graph $G' = (V - \{v\}, E \cap \{\{u, w\} \mid u, w \in V \wedge u \neq v \wedge w \neq v\})$ is connected for every $v \in V$. □

Bi-connected graphs have an important well known property which we exploit further. Each bi-connected graph can be constructed starting with a cycle by an operation of *adding loop (handle)* to the graph (Tarjan, 1972; West, 2000).

Adding a loop which is a sequence of vertices $L = [u, x_1, x_2, ..., x_l, v]$ to an undirected graph $G = (V, E)$ where $u, v \in V$ and $x_i \notin V$ for $i = 1, 2, \ldots, l$ (x_i are new vertices) means to create a new graph $G' = (V', E')$; where $V' = V \cup \{x_1, x_2, \ldots, x_l\}$ and either $E' = E \cup \{\{u, v\}\}$ in the case when $l = 0$ or $E' = E \cup \{\{u, x_1\}, \{x_1, x_2\}, \ldots, \{x_{l-1}, x_l\}, \{x_l, v\}\}$ in the case when $l \geq 1$. As a preparation for the design of algorithms, the loop L is assigned a cycle $C(L)$ if the graph G is connected. The cycle $C(L)$ consists of vertices on a path between u and v in G followed by vertices $x_1, x_2, \ldots, x_l$. Let us call the above construction sequence of the bi-connected graph a *loop decomposition*.

Lemma 1 (loop decomposition) **(Tarjan, 1972; West, 2000).** Any bi-connected graph $G = (V, E)$ can be obtained from a cycle by the **operation of adding a loop**. Moreover, the corresponding loop decomposition can be effectively found in the worst case time of $O(|V| + |E|)$. ■

Due to the inductive character of the lemma 1, observe that the currently constructed graph is bi-connected at any stage of the construction.

Optimal Macros in Bi-connected Graphs

We are about to exploit certain kind of a *solution database* containing pre-calculated optimal solutions to special sub-problems. Our concept of solution database is similar to the common concept of pattern database (Culberson & Schaeffer, 1996). The difference is that instead of storing values of a heuristic function we are storing the complete solutions.

The structurally simplest solvable sub-problem of the pebble motion problem consists of a so-called θ-like graph (see figure 2) where there is a single unoccupied vertex (Surynek, 2009b).

Definition 5 ***(θ-like graph).*** Let $A = \{x_1, x_2, \ldots, x_a\}$, $B = \{y_1, y_2, \ldots, y_b\}$, and $C = \{z_1, z_2, \ldots, z_c\}$ be finite sets (of vertices) where $|A| \geq 1 \wedge |B| \geq 2 \wedge |C| \geq 1$. A *θ-like graph* $G_\theta(A, B, C) = (V_\theta, E_\theta)$ is an undirected graph where $V_\theta = A \cup B \cup C$ and $E_\theta = \{\{x_1, x_2\}, \ldots, \{x_{a-1}, x_a\}, \{y_1, y_2\}, \ldots, \{y_{b-1}, y_b\}, \{z_1, z_2\}, \ldots, \{z_{c-1}, z_c\}, \{x_1, y_1\}, \{x_a, y_b\}, \{y_1, z_1\}, \{y_b, z_c\}\}$. □

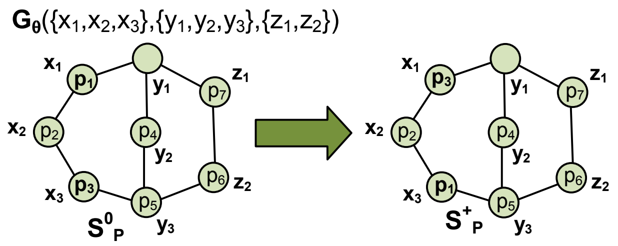


Figure 2: *An example of θ-like graph.* The task is to transpose pebbles p_1 and p_3.

The number of all the possible θ-like graphs grows as polynomial with respect to the number of vertices (namely they are $O(|V_\theta|^3)$). However, the number of all the possible pebble motion problems on θ-like graphs grows exponentially with respect to the number of vertices (they are proportional to the number of permutations of the set of vertices). Hence, a restriction on the number of problems whose solution will be stored in the solution database must be made. Additionally, we need such problems from which a solution to the general problem can be composed. The following cases of problems satisfy both.

In the following text, we suppose (without loss of generality) that the unoccupied vertex in the initial and the goal arrangements of pebbles in θ-like graphs is the vertex y_1.

Definition 6 ***(transposition case).*** Let $G_\theta(A,B,C)$ be a θ-like graph and let $P=\{p_1,p_2,\ldots,p_\mu\}$ be a set of pebbles with $\mu=|V_\theta|-1$. The pebble motion problem on a graph with the initial arrangement S_P^0 and the goal arrangement S_P^+ is called a *transposition case*, if there are pebbles $p,q\in P$ such that $p\neq q$ and $S_P^0(p)=S_P^+(q)$, $S_P^0(q)=S_P^+(p)$, and $(\forall r\in P)(r\neq p,q\Rightarrow S_P^0(r)=S_P^+(r))$ (see figure 2). □

Definition 7 ***(3-cycle rotation case).*** Let $G_\theta(A,B,C)$ be a θ-like graph and let $P=\{p_1,p_2,\ldots,p_\mu\}$ be a set of pebbles with $\mu=|V_\theta|-1$. The pebble motion problem on a graph with the initial arrangement S_P^0 and the goal arrangement S_P^+ is called a *3-cycle rotation case*, if there are pebbles $p,q,s\in P$ such that p, q, s are pair-wise distinct and $S_P^0(p)=S_P^+(q)$, $S_P^0(q)=S_P^+(s)$, $S_P^0(s)=S_P^+(p)$, and $(\forall r\in P)(r\neq p,q,s\Rightarrow S_P^0(r)=S_P^+(r))$ (see figure 2). □

Both, the number of transposition cases as well as the number of 3-cycle rotation cases, grow polynomially with respect to the number of vertices (they are $O(|V_\theta|^5)$ and $O(|V_\theta|^6)$ respectively; three numbers are necessary for identifying a θ-like graph and two or three numbers are necessary to identify vertices involved in the transposition or the 3-cycle rotation case respectively). Thus it is realistic to store all the optimal solutions (macros) of the described cases up to the certain size of θ-like graphs in the solution database.

The following two lemmas summarize usefulness of the transposition case and the 3-cycle rotation case for solving the general problem.

Lemma 2 (solvability – transposition case) **(Wilson, 1974).** A transposition case of the pebble motion problem on a θ-like graph $G_\theta(A,B,C)$ with $|A|\neq 2\vee|B|\neq 3\vee |C|\neq 2$ is solvable, if and only if G_θ contains a **cycle of** the **odd length**. A solution to **any problem** of pebble motion on a θ-like graph $G_\theta(A,B,C)=(V_\theta,E_\theta)$ can be composed of at most $|V_\theta|-2$ solutions to **transposition cases** in the same graph. Moreover, a sequence of transposition cases whose solutions are necessary for producing the overall solution can be determined in the worst case time of $O(|V_\theta|)$. ■

The goal arrangement of robots S_P^+ in a θ-like graph $G_\theta(A,B,C)=(V_\theta,E_\theta)$ can be regarded as a permutation over $|V_\theta|-1$ elements with respect to the initial arrangement S_P^0. S_P^+ represents an *even permutation* with respect to S_P^0, if it is reachable using the even number of solutions to transposition cases. Otherwise it represents an *odd permutation*.

Lemma 3 (solvability – 3-cycle case) **(Kornhauser *et al.*, 1984).** A 3-cycle rotation case of the problem of pebble motion on a θ-like graph $G_\theta(A,B,C)$ with $|A|\neq 2\vee |B|\neq 3\vee|C|\neq 2$ is **always solvable**. A solution to any pebble motion problem whose goal arrangement of pebbles S_P^+ represents an **even permutation** with respect to the initial arrangement S_P^0 in a θ-like graph $G_\theta(A,B,C)=(V_\theta,E_\theta)$ can be composed of at most $|V_\theta|-2$ solutions to **3-cycle rotation cases** in the same graph. Moreover, a sequence of 3-cycle rotation cases necessary for the task can be effectively determined in the worst case time of $O(|V_\theta|)$. ■

The exception of $G_\theta(A,B,C)$ with $|A|=2\wedge|B|=3\wedge |C|=2$ can be solved separately. Due to small size of this exception, solutions to **all** the problems over this graph can be pre-calculated into the solution database (that is, solutions for all permutations of pebbles are stored).

At this point, we know how to solve the general pebble motion problem on a θ-like graph by composing a solution of the macros for transposition and 3-cycle case. Let us now further generalize the approach for all the bi-connected graphs.

A covering of the given bi-connected graph with θ-like sub-graphs is the first step. That is, a set of θ-like graphs $\theta_1,\theta_2,\ldots,\theta_t$ such that $G=\bigcup_{i=1}^t\theta_i$ is needed. Let us call this covering a *θ-decomposition* of the bi-connected graph. If such θ-decomposition is available, then the remaining question is how to move robots to their target θ-like sub-graphs of the θ-decomposition. Goal positions of robots within θ-like sub-graphs can be then reached using macros from the solution database. The following lemmas justify the existence of θ-decomposition of the bi-connected graph.

Lemma 4 (two disjoint paths) **(West, 2000).** Let $G=(V,E)$ be a bi-connected graph and let $u,v\in V$ be two distinct vertices. There exist **two vertex disjoint paths** between u and v. Moreover, these two path can be effectively determined in the worst case time of $O(|V|+|E|)$. ■

Lemma 5 (θ-decomposition). Let $G=(V,E)$ be a bi-connected graph not being a single cycle. Then there **exists** a **θ-decomposition** $\theta_1,\theta_2,\ldots,\theta_t$ (θ_i is a θ-like graph for $i=1,2,\ldots,t$) such that $G=\bigcup_{i=1}^t\theta_i$. Moreover, the θ-decomposition of the graph can be effectively found in $O(|V|+|E|)$. ■

Proof. From lemma 1, we know that there exists a loop decomposition of the bi-connected graph G. Consider the last loop $L=[u,x_1,x_2,\ldots,x_l,v]$ of the loop decomposition. The graph G without the loop L is again a bi-connected graph, let us denote it G^-. Using lemma 4, there exist two vertex disjoint paths π,ψ connecting u and v in G^-. Now $G_\theta(\pi,L,\psi)$ is the θ-like graph. Inductively suppose to have a θ-decomposition of G^-. Together with $G_\theta(\pi,L,\psi)$ we have the θ-decomposition of G. ■

Solving Algorithms for Bi-connected Case

Two algorithms for solving pebble motion problems on a bi-connected graph $G=(V,E)$ with a single unoccupied vertex ($\mu=|V|-1$) are presented below. Both algorithms assume that a loop decomposition of the graph G was constructed. That is, we have a cycle C_0 and a sequence of loops $L_1,L_2,\ldots,L_t$ such that the graph G can be constructed from C_0 by adding loops $L_1,L_2,\ldots,L_t$ incrementally. If the graph G contains a cycle of odd length, C_0 is also supposed to be of odd length. Since the construction of the graph G starts with a cycle C_0 (which is a connected graph) $C(L_i)$ is defined for every $i = 1,2,\ldots,t$. Specially, we define $C(C_0) = C_0$.

To reduce the complexity of the pseudo-code of algorithms we assume the unoccupied vertex of the goal arrangement S_P^+ to be in the cycle C_0 (that is, $(v \in V \wedge (\forall p \in P) S_P^+(p) \neq v) \Rightarrow v \in C_0$). Overcoming this assumption is discussed in the next section.

Except the functions S_P^0 and S_P^+ we further have a function $S_P : P \rightarrow V$ expressing current positions of pebbles. Next, we have functions $\Phi_P^0 : V \rightarrow P \cup \{\perp\}$, $\Phi_P^+ : V \rightarrow P \cup \{\perp\}$, and $\Phi_P : V \rightarrow P \cup \{\perp\}$ which are generalized inverses of S_P^0, S_P^+, and S_P respectively; the symbol $\perp$ stands for unoccupied vertex (that is, $(\forall p \in P)\Phi_P(S_P(p)) = p$; $\Phi_P(v) = \perp$ if $(\forall p \in P) S_P(p) \neq v$). Next, we assume that we have a sequence of potentially infinite sequences representing the solution of the problem $[M_{p_1}, M_{p_2}, \ldots, M_{p_\mu}]$.

An Algorithm Based on θ-decomposition

In this section, we describe an improvement of the solving algorithm from (Kornhauser *et al.*, 1984).

Algorithm 1. *The MIT-θ algorithm*. The algorithm solves a given pebble motion problem on a bi-connected graph modeling the environment with a single unoccupied vertex.

```
function MIT-θ-Solve ($G, S_P^0, S_P^+$) : pair
1: $\zeta \leftarrow 0$; $S_P \leftarrow S_P^0$
2: for $c = t, t-1, \ldots, 2$ do
3:   if $|L_c| > 2$ then
4:     SolveRegular-$\theta(c)$
5: let $[u, x_1, x_2, \ldots, x_l, v] = L_1$
6: let $\pi, \psi$ be two disjoint paths between
7:   $u$ and $v$ in $C_0$
8: θ-BOX-Solve ($G_\theta(\pi, L_1, \psi), S_P, S_P^+$)
9: return ($\zeta, [M_{r_1}, M_{r_2}, \ldots, M_{r_\mu}]$)
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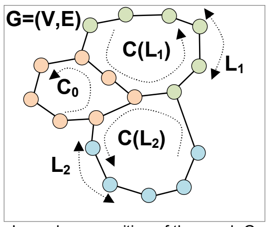

Loop decomposition of the graph G

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procedure SolveRegular-θ ($c$)
1: let $[u, x_1, x_2, \ldots, x_l, v] = L_c$
2: lock ($L_c$) ; unlock ($\{u, v\}$)
3: let $\pi, \psi$ be two disjoint paths between $u$
4:   and $v$ not containing locked vertices
5: let $(V_\theta, E_\theta) = G_\theta(\pi, L_c, \psi)$
6: for $i = 1,2,\ldots,l$ do
7:   if $S_P(\Phi_P^+(x_i)))) \notin V_\theta$ then
8:     lock ($L_c$) ; unlock ($\{u, v\}$)
9:     MovePebble ($\Phi_P^+(x_i), v$)
10:    MoveUnoccupied ($u$)
11:    unlock ($L_c$)
12:    $S_\theta^+ \leftarrow S_P$; $S_\theta^+(\Phi_P^+(v)) = S_P^+(\Phi_P^+(v))$
13:    θ-BOX-Solve ($G_\theta(\pi, L_1, \psi), S_P, S_\theta^+$)
14:  else
15:    lock ($L_c$) ; unlock ($\{u, v\}$)
16:    MoveUnoccupied ($u$)
17:    unlock ($L_c$)
18:    $S_\theta^+ \leftarrow S_P$; $S_\theta^+(\Phi_P^+(x_i)) = S_P^+(\Phi_P^+(x_i))$
19:    θ-BOX-Solve ($G_\theta(\pi, L_1, \psi), S_P, S_\theta^+$)
20:  lock ($L_c$) ; unlock ($\{u, v\}$)
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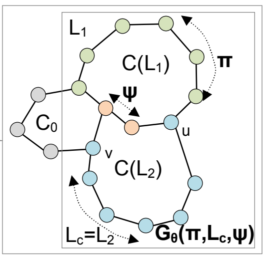
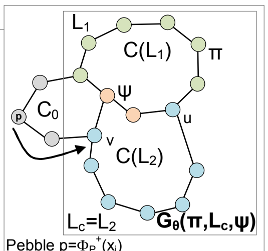
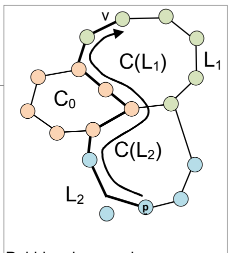




Pebble p=$\Phi_P^+(x_i)$ outside Gθ; move p to v

```
procedure MoveUnoccupied ($v$)
1: let $x \in V$ such that $\Phi_P(x) = \perp$ and $x$ is not locked
2: let $[x = k_1, k_2, \ldots, k_j = u]$ be a shortest path between
3:   $x$ and $v$ in $G$ not containing locked vertices
4: for $i = 1,2,\ldots,j-1$ do
5:   SwapPebblesUnoccupied ($k_{i+1}, k_i$)
```

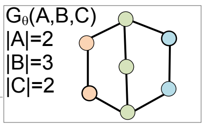
```
procedure MovePebble ($p, v$)
1: let $[S_P(p) = k_1, k_2, \ldots, k_j = v]$ be a shortest path between $S_P(p)$ and $v$
2:   in $G$ not containing locked vertices
3: for $i = 1,2,\ldots,j-1$ do
4:   lock ($\{k_i\}$)
5:   MoveUnoccupied ($k_{i+1}$)
6:   unlock ($\{k_i\}$)
7:   SwapPebbleUnoccupied ($k_i, k_{i+1}$)
```

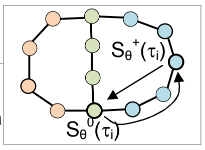


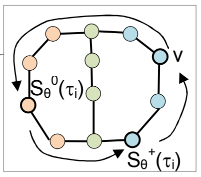
Pebble p is moved to v by rotating cycle C(L2), C0, and C(L1)

```
procedure SwapPebblesUnoccupied ($u, v$)
1: $S_P(\Phi_P(u)) = v$; $p = \Phi_P(u)$
2: $\Phi_P(u) = \perp$; $\Phi_P(v) = p$
3: for $i = 1,2,\ldots,\mu$ do
4:   $m_\zeta^{p_i} = S_P(p_i)$
5: $\zeta \leftarrow \zeta + 1$
```

```
procedure θ-BOX-Solve ($G_\theta(A, B, C), S_\theta^0, S_\theta^+$)
1: let $(V_\theta, E_\theta) = G_\theta(A, B, C)$
2: let $\{\tau_1, \tau_2, \ldots, \tau_{|V_\theta|-1}\} = \{\tau \mid S_\theta^0(\tau) \in V_\theta\}$
3: if $|A| = 2 \wedge |B| = 3 \wedge |C| = 2$ then
4:   ApplyMacro ($table^{232}[S_\theta^0, S_\theta^+]$)
5: else
6:   $S_\theta \leftarrow S_\theta^0$
7:   if $G_\theta$ contains an odd cycle then
8:     for $i = 1,2,\ldots,|V_\theta| - 2$ do
9:       if $S_\theta(\tau_i) \neq S_\theta^+(\tau_i)$ then
10:        ApplyMacro ($table_T^{G_\theta}[S_\theta(\tau_i), S_\theta^+(\tau_i)]$)
11:  else { $G_\theta$ does not contain any odd cycle}
12:    if $S_\theta^+$ gives an odd permutation w.r.t. $S_\theta$ then
13:      fail {the problem is unsolvable}
14:    else { $S_\theta^+$ gives an even permutation w.r.t. $S_\theta$ }
15:      for $i = 1,2,\ldots,|V_\theta| - 2$ do
16:        if $S_\theta(\tau_i) \neq S_\theta^+(\tau_i)$ then
17:          let $v \neq S_\theta(\tau_i), S_\theta^+(\tau_1), S_\theta^+(\tau_2), \ldots, S_\theta^+(\tau_i)$
18:          ApplyMacro ($table_3^{G_\theta}[S_\theta(\tau_i), S_\theta^+(\tau_i), v]$)
```







```
procedure ApplyMacro ($\sigma$)
1: let $[(u_1, v_1), (u_2, v_2) \ldots, (u_k, v_k)] = \sigma$
2: for $i = 1,2,\ldots,k$ do
3:   SwapPebblesUnoccupied ($u_i, v_i$)
4:   $S_\theta(\Phi_P(u)) = v$
```

The solving process of 3-cycle case that originally exploits 3-transitivity of θ-like sub-graphs is replaced by the use of macros. The resulting algorithm is called *MIT-θ* and it is formalized below using the pseudo-code as algorithm 1 (the original version of the algorithm from (Kornhauser *et al.*, 1984) is called *MIT*). The solving algorithm itself is represented by the function *θ-BOX*-Solve with several auxiliary functions. Next, there is a procedure *θ-BOX*-Solve which represents the solving process within θ-like graphs using pre-calculated optimal macros from the solution database.

The solving algorithm proceeds inductively according to the pre-calculated loop decomposition $L_1, L_2, \ldots, L_t$ (lines 2-4 of *MIT-θ*-Solve). The pebbles are placed to their goal positions in loops starting with the last loop L_t and continuing to the original cycle with the loop (C_0, L_1 - original θ-like graph; lines 5-8 of *MIT-θ*-Solve). Having a loop L_c of the loop decomposition, a corresponding θ-like subgraph is considered (lemma 5; lines 1-5 of SolveRegular-θ). All the robots whose goal positions are within the loop are placed. Two cases are distinguished. If the pebble to be placed is already within the θ-like sub-graph, then a macro is used to place it to the right position (lines 14-19 of SolveRegular-θ). If the pebble is outside the θ-like sub-graph, then it must be first moved to into the θ-like sub-graph before the macro can be applied (lines 7-13 of SolveRegular-θ). The original cycle C_0 with its loop L_1 is solved solely using macros (lines 5-8 of *MIT-θ*-Solve), since all the pebbles whose goal positions are within the original θ-like sub-graph are already there.

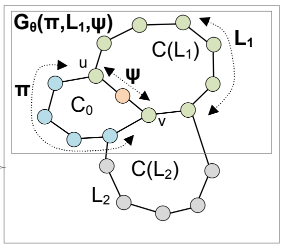
Without proof, let us summarize properties of the algorithm. The *MIT-θ* algorithm is **sound** and **complete**. The worst case time complexity is of $O(|V|^5)$.

An Algorithm Exploiting Loop Decomposition

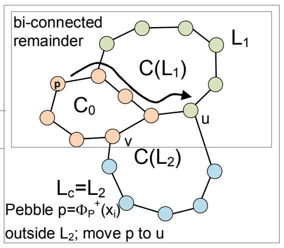
The second algorithm for solving pebble motion problems on bi-connected graphs is called *BIBOX-θ*.

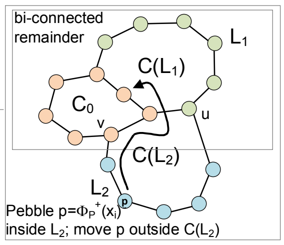
Algorithm 2. *The BIBOX-θ algorithm.* The algorithm solves a given pebble motion problem on a bi-connected graph modeling the environment with a single unoccupied vertex.

```
function BIBOX-θ-Solve $(G, S_P^0, S_P^+)$ : pair
1: $\zeta \leftarrow 0$ ; $S_P \leftarrow S_P^0$
2: for $c = t, t-1, \dots, 2$ do
3:   if $|L_c| > 2$ then
4:     SolveRegularCycle $(c)$
5: let $[u, x_1, x_2, \dots, x_l, v] = L_1$
6: let $\pi, \psi$ be two disjoint paths between
7:   $u$ and $v$ in $C_0$
8: θ-BOX-Solve $(G_\theta(\pi, L_1, \psi), S_P, S_P^+)$
9: return $(\zeta, [M_{p_1}, M_{p_2}, \dots, M_{p_\mu}])$
```

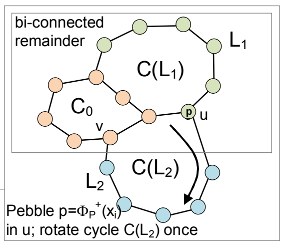


```
procedure SolveRegularCycle $(c)$
1: let $[u, x_1, x_2, \dots, x_l, v] = L_c$
2: for $i = 1, 2, \dots, l$ do
3:   if $S_P(\Phi_P^+(x_i)) \notin L_c$ then
4:     lock $(L_c)$ ; unlock $(\{u, v\})$
5:     MovePebble $(\Phi_P^+(x_i), u)$
6:     MoveUnoccupied $(v)$
7:     unlock $(L_c)$
8:     RotateCycle$^+$ $(C(L_c))$
9:   else
10:    lock $(L_c)$ ; unlock $(\{u, v\})$
11:    MoveUnoccupied $(u)$
12:    unlock $(L_c)$
13:    $\rho \leftarrow 0$
14:    while $S_P(\Phi_P^+(x_i))) \neq v$ do
15:      RotateCycle$^+$ $(C(L_c))$
16:      $\rho \leftarrow \rho + 1$
17:    lock $(L_c)$ ; unlock $(\{u, v\})$
18:    let $o \in V - (\bigcup_{i=c}^{k} L_i \cup C(L_c))$
19:    MovePebble $(\Phi_P^+(x_i), o)$
20:    lock $(\{o\})$
21:    MoveUnoccupied $(u)$
22:    unlock $(L_c)$
23:    while $\rho > 0$ do
24:      RotateCycle$^-$ $(C(L_c))$
25:      $\rho \leftarrow \rho - 1$
26:    unlock $(\{o\})$
27:    MovePebble $(\Phi_P^+(x_i), u)$
28:    MoveUnoccupied $(v)$
29:    RotateCycle$^+$ $(L_c)$
30: lock $(L_c)$ ; unlock $(\{u, v\})$
```





bi-connected remainder
L_1
$C(L_1)$
C_0
v
u
$C(L_2)$
L_2
p
Pebble $p=\Phi_P^+(x_i)$ inside L_2; move p outside $C(L_2)$



```
procedure RotateCycle$^+$ $(C)$
1: let $x \in C$ such that $\Phi_P(x) = \perp$ and $x$ is not locked
2: for $i = 1, 2, \dots, |C|$ do
3:   SwapPebblesUnoccupied $(prev/V(C, x), x)$
4:   $x \leftarrow prev/V(C, x)$
```

It is a modification of the algorithm from (Surynek, 2009a) (the original algorithm is called *BIBOX*) where the last phase of the algorithm placing the pebbles in the original cycle C_0 is replaced by solving process over the corresponding θ-like sub-graph. The main contribution of this approach is that now we need only **one** unoccupied vertex while the original version of the algorithm requires at least **two** unoccupied vertices.

For easier expressing of the algorithm we have auxiliary functions $next/V(C, v)$, $prev/V(C, v)$ that return the next or the previous vertex in the given cycle with respect to the clock-wise orientation of the cycle. The solving algorithm itself is presented here using pseudo-code as algorithm 2.

The algorithm proceeds from the last loop to the first loop of the loop decomposition. This process is very similar to the corresponding process within the *MIT-θ* algorithm. The main difference rests in a manner how the pebbles are placed within a loop. Within a loop, pebbles are placed to their goal positions in the stack manner (that is, a new pebble comes at the beginning of the loop and the loop is rotated - stack pushes). The last rotation of the loop places the pebbles to their destinations. When placing a pebble within the loop it is necessary to distinguish between the situation when the pebble is outside the loop (lines 3-8 of SolveRegularCycle) and the situation when the pebble is already within the current loop (lines 10-29 of SolveRegularCycle).

Again without proof, let us summarize properties of the algorithm. The *BIBOX-θ* algorithm is **sound** and **complete**. The worst case time complexity of the algorithm is $O(|V|^4)$.

Extensions and the Real Implementation

The presented pseudo-codes of the *MIT-θ* and the *BIBOX-θ* algorithms require a special assumption that the finally unoccupied vertex must be in the original cycle. To overcome this assumption we need to modify the required solution given by the function S_P^+ so that unoccupied vertex is moved to the original cycle along a path π. After solving the problem by the algorithm the unoccupied vertex is moved back along the path π which finishes the solution of the original unmodified problem.

If the required record is not in the solution database, then the algorithm should switch to solving method based on 4-transitivity from (Kornhauser *et al.*, 1984).

Solving Multi-robot Path Planning Problems

Having a solving algorithm for the pebble motion problem on a graph, it is easy to solve the corresponding multi-robot path planning problem. We can just proclaim the solution of the pebble motion problem to be a solution of the corresponding multi-robot path planning problem. However, this may waste parallelism.

The more sophisticated approach is to utilize the relaxed requirements on the solution in the multi-robot path planning problem to increase parallelism. The method of choice here is *critical path* (Russell & Norvig, 2003). We define a relation of dependence between motions of pebbles. Two motions are dependent if one must precede the other in the solution (for example two consecutive motions of the same pebble are dependent). The (anti-symmetric) relation of dependence induces a directed acyclic graph on the set of vertices represented by moves. The method of critical path can be used to calculate earliest time step for each move when it can be executed. A deeper discussion about this approach is given in (Surynek, 2009c).

Experimental Evaluation

The presented algorithms - MIT-θ and BIBOX-θ as well as its competitors - were implemented in C++ and an experimental evaluation was made. The experimental evaluation was made on a machine with Pentium 4 2.4 GHz with 512Mb of memory under Mandriva Linux 10.1. Source code and additional data for reproducing all the experiments are available at: http://ktiml.mff.cuni.cz/~surynek/research/icaps2009/. The comparison was concentrated on the length of solutions and on the solving runtime. The results are presented in figure 3 and 4.

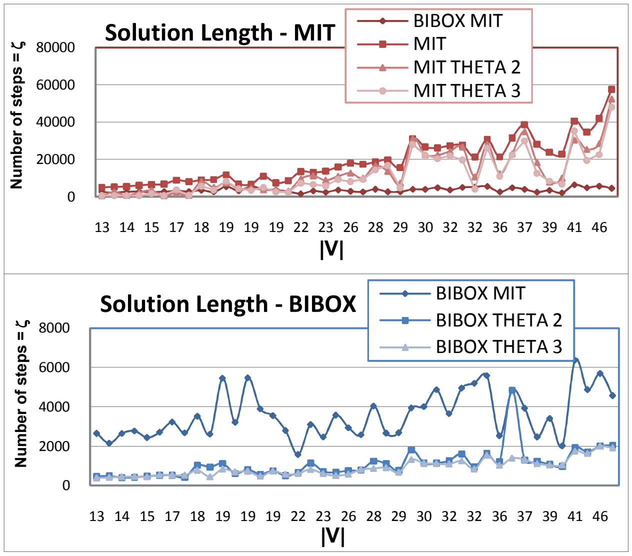


Figure 3. *Solution length comparison.* Six variants of solving algorithms are compared – *BIBOX* where the solving process for original cycle with loop is based on 4-transitivity (BIBOX MIT), the original *MIT* algorithm, *MIT-θ* where transposition cases are preferably used (MIT THETA 2), *MIT-θ* where 3-cycle rotations are preferably used (MIT THETA 3), *BIBOX-θ* where transposition cases are preferably used (BIBOX THETA 2), and *BIBOX-θ* where 3-cycle rotations are preferably used (BIBOX THETA 3).

In all the tests, necessary optimal macros were found in the database (that is, the alternative method based on 4-transitivity was not used). The results show that replacement of the method based on 4-transitivity with optimal macros brings significant improvement in solution length and solving time of both algorithms. Moreover, the experiments show that all the variants of the *BIBOX* algorithm outperform the *MIT* algorithm significantly. It is also evident that the preference of 3-cycle rotation cases is slightly better than the preference of transposition case with respect to the solution length. However, notice that storing transposition cases in the solution database is less space consuming.

The tests were made on random instances of problems of pebble motion on bi-connected graphs where the number of vertices ranged from 13 to 48. The number of loops of the loop decomposition ranged from 3 to 16. The length of loops of the decomposition had the random length with the uniform distribution in the interval of 1…8. All the problems had a single unoccupied vertex placed randomly (generally not in the original cycle).

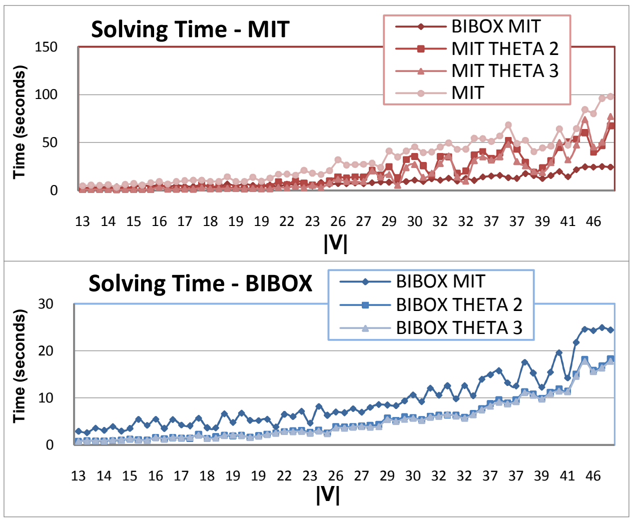


Figure 4. *Solving time comparison.* Six variants of solving algorithms are compared – see figure 3. Each problem was solved 1000 time to accumulate measurable time.

Related Works and Conclusions

This work is significantly influenced by (Ryan, 2007). The author presents a solving method for the multi-robot path planning based on a decomposition of the environment into simpler sub-graphs that are easier to tackle. This approach has much in common with the approach presented above. However, deep theoretical results gained for pebble motion on graphs (sliding box puzzles) (Wilson, 1974; Kornhauser *et al.*, 1984; Ratner & Warmuth, 1986) are ignored in (Ryan, 2007), though they are so closely related to multi-robot path planning.

The major aim of this paper is to fill in the gap between theory and practical solving of problems of pebble motion on graph and multi-robot path planning. We have to emphasize that this paper intensively builds on existing works while we improve aspects regarding the optimality of solutions.

Let us further comment the related works. Graph theoretical properties crucial for tackling the problem were identified in (Wilson, 1974; Kornhauser *et al.*, 1984). The solving methods for transposition and 3-cycle rotation cases ware developed in (Surynek, 2009b). The less general version of the *BIBOX* algorithm is presented in (Surynek, 2009a). This version of the *BIBOX* algorithm requires at least two unoccupied vertices in a bi-connected graph.

A comparison with domain-independent planners and scaling evaluation is also given in (Surynek, 2009b) (*LPG-td* and *SGPlan* were tested; only extremely small pebble motion/multi-robot problems are solvable by domain-independent planners). These results render the domain-independent approach to be uncompetitive.

Another interesting approach for solving the problem of multi-robot path planning is represented by works (Silver, 2005) and (Wang & Botea, 2009). They both build upon the standard *A** search (Russell & Norvig, 2003) while the efficiency is increased by searching a path to the destination for each robot independently if it is possible. Clearly, this approach becomes problematic on problems with few unoccupied vertices where it is impossible to find a path for a robot independently on other robots. In contrast to this, our approach works even on extremely crowded graphs and does not use any search (which may consume exponential time).

Our work can be summarized as follows. A successful application of optimal pre-calculated macros for solving problems of pebble motion and multi-robot path planning with bi-connected environments has been presented in this paper. One existing algorithm (*MIT*) was improved by the integration of macros. Another algorithm (*BIBOX*) was improved and generalized – the new variant is called ***BIBOX-θ*** - so that is becomes one of the best algorithms (it is better than existing domain-dependent algorithms as well as domain independent planners) for solving the studied class of problem in terms of runtime and the quality of solutions. For future work we plan to develop techniques for post-processing solutions produced by the presented algorithms. The post-processing will be aimed on shortening solutions and increasing parallelism.

Acknowledgement

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